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Abstract

In the *m-Peripatetic Salesman Problem* (m-PSP) the aim is to determine m edge disjoint Hamiltonian cycles of minimum total cost on a graph. This article introduces new valid inequalities and polyhedral results for the m-PSP. An improved 2-index branch-and-cut algorithm is developed. Tests performed on randomly generated and TSPLIB Euclidean instances indicate that this algorithm can solve instances with more than double size of what was previously achievable.
Key Words: Peripatetic Salesman Problem, Traveling Salesman Problem, Branch-and-Cut, Facets.

Résumé

Dans le problème du voyageur de commerce m-péripatétique (m-PSP) on cherche à déterminer m cycles hamiltoniens n'ayant aucune arête en commun et dont la somme des coûts des arêtes est minimale. Cet article introduit de nouvelles inégalités valides et des résultats polyhédraux pour le m-PSP. Un Branch-and-Cut amélioré s'appuyant sur un modèle à 2 indices est développé. Les tests appliqués à des instances Euclidiennes aléatoirement générées et aux instances de TSPLIB indiquent que cet algorithme peut résoudre des instances de taille deux fois plus grande que celles qui étaient résolues avec les algorithmes précédents.
Mots Clefs: Problème du voyageur m-péripatétique, problème du voyageur de commerce, branch-and-cut, facettes.

1 Introduction

The undirected *m-Peripatetic Salesman Problem* (m-PSP) is defined on a complete graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ is the vertex set and $E = \{e = (i, j) : i, j \in V, i < j\}$ is the edge set. With each edge e is associated a cost c_e. The m-PSP consists of determining m edge disjoint Hamiltonian cycles of least total cost on G. The m-PSP is NP-hard and reduces to the *Traveling Salesman Problem* (TSP) when $m = 1$. To avoid trivial or infeasible cases we assume that $n \geq 6$ and $m \leq \lfloor (n-1)/2 \rfloor$. The m-PSP was introduced by Krarup [9] and has applications in the construction of watchman tours [15] and in the design of robust networks in the presence of edge failures [5]. There exist relatively few studies on the m-PSP. In his seminal paper, Krarup [9] presented a simple procedure to determine an upper bound $\overline{z}$ on the optimal m-PSP solution value z^*. Initially $F = \emptyset$ and $\overline{z} = 0$. The procedure executes the following iteration m times: solve a TSP on $E \backslash F$ and let z be its optimal value; set $\overline{z} := \overline{z} + z$ and $F := F \cup \{e \in E : e \text{ belongs to the TSP solution}\}$. Duchenne, Laporte and Semet [6] have shown that on random instances with $n \leq 50$, this upper bound (called the *Krarup bound*) typically lies within 2.3% of z^*. De Kort [4] has derived a lower bound based on the solution of a *Capacitated Transportation Problem*. When $m = 2$ this bound lies on average within 8.9% of the Krarup bound. De Kort [5] also developed a branch-and-bound algorithm for the 2-PSP which was applied to the solution of random ($40 \leq n \leq 130$) and Euclidean ($40 \leq n \leq 60$) instances. Duchenne, Laporte and Semet [6] have formulated the m-PSP by means of a 3-index formulation and have derived from it a 2-index formulation which constitutes a relaxation of the problem. Some valid inequalities were proposed for each formulation and two branch-and-cut algorithms were developed and successfully applied to the solution of random ($10 \leq n \leq 60$; $2 \leq m \leq 5$) and Euclidean ($10 \leq n \leq 50$; $2 \leq m \leq 5$) instances.

The purpose of this article is to extend the Duchenne, Laporte and Semet [6] contribution in two directions. First, new valid inequalities are introduced for each formulation, and polyhedral results are derived. Second, a new branch-and-cut algorithm employing these inequalities is developed and shown to be superior to the existing algorithms. In order to introduce these results it will be necessary to recall some of the material presented in [6]. We have, however, attempted to minimize the overlap between the two papers and to clearly distinguish the new contribution from the old.

The remainder of this article is organized as follows. In Section 2 we restate the 3-index model and present a new class of valid inequalities. Polyhedral results associated with this model are developed. In Section 3 we introduce new valid inequalities for the 2-index model, some polyhedral results, and projected inequalities which are valid for the m-PSP. Algorithms are developed in Section 4, followed by computational results in Section 5, and by conclusions in Section 6.

2 3-index formulation

In [6], the 3-index model is defined with binary variables x_{ijk} equal to 1 if and only if edge (i,j) appears in tour k. Here we use the more compact notation x_{ek}, where $e=(i,j)$, and c_e instead of c_{ij}. For $S \subset V$, define $E(S)=\{e=(i,j): i,j \in S, i<j\}$, $\delta(S)=\{e=(i,j): i \in S, j \in V \setminus S$ or $i \in V \setminus S, j \in S, i<j\}$, and $\delta(i)=\delta(\{i\})$. The 3-index formulation is then:

$$\text{(3-index)} \qquad \text{minimize} \sum_{k=1}^{m} \sum_{e \in E} c_e x_{ek} \tag{1}$$

subject to:

$$\sum_{e \in \delta(i)} x_{ek} = 2 \qquad (i \in V,\ k=1,\ldots,m) \tag{2}$$

$$\sum_{e \in E(S)} x_{ek} \leq |S|-1 \qquad (S \subset V,\ 3 \leq |S| \leq \lfloor n/2 \rfloor,\ k=1,\ldots,m) \tag{3}$$

$$\sum_{k=1}^{m} x_{ek} \leq 1 \qquad (e \in E) \tag{4}$$

$$x_{ek} \in \{0,1\} \qquad (e \in E,\ k=1,\ldots,m). \tag{5}$$

Constraints (2), (3) and (5) of this formulation are the degree, subtour elimination and integrality constraints of the TSP for each k (see, e.g., [3]). Constraints (4) simply state that an edge can appear in only one Hamiltonian cycle.

Because constraints (2)-(5) define m edge disjoint TSP solutions on G, any valid TSP inequality defined in terms of x_e variables (see, e.g. , [7]) is also valid for 3-index when defined in terms of the x_{ek} variables. Valid constraints for 3-index include 2-matching inequalities, combs inequalities and clique tree inequalities. The following results show that some polyhedral properties applicable to the TSP extend to the m-PSP.

Lemma 1. *Let $G=(V,E)$ be a complete graph with $V=\{1,\ldots,n\}$ and let $m \leq \lfloor (n-1)/2 \rfloor$. Then, given a Hamiltonian cycle C on G, there always exists an m-PSP solution containing C as one of its cycles.*

Proof. Walecki, quoted by Lucas in [10], proves that a complete graph with n vertices admits a Hamiltonian decomposition, i.e., a decomposition into $\lfloor (n-1)/2 \rfloor$ edge disjoint Hamiltonian cycles and, if necessary, a perfect matching. Without loss of generality, we can assume that C is the first Walecki decomposition cycle and the thesis follows. □

Let $\mathcal{C}_{PSP}$, $\mathcal{C}_{PSP'}$ and $\mathcal{C}_{TSP}$ denote the convex hull of the set of feasible solutions of the m-PSP (defined by (2)-(5)), of the m-PSP relaxation (defined by (2), (3) and (5)), and of the TSP (defined by (2)-(5) and $m=1$), respectively.

Theorem 1. *$dim(\mathcal{C}_{PSP}) = mn(n-5)/2$.*

Proof. Since the dimension of $\mathcal{C}_{TSP}$ is $n(n-3)/2$, the dimension of $\mathcal{C}_{PSP}$ is $mn(n-3)/2$. Constraints (4) are satisfied as equalities for all e such that the edge e is in the m-PSP solution. Therefore, every solution contains mn contraints (4) satisfied as equalities. Thus $dim(\mathcal{C}_{PSP}) = mn(n-3)/2 - mn = mn(n-5)/2$. □

Let $x \in \mathbb{R}^{mn(n-1)/2}$ be a vector describing a feasible solution to (2)-(5), and let $x^k \in \mathbb{R}^{n(n-1)/2}$ be a vector corresponding to the k^{th} feasible cycle.

Theorem 2. *If $F = \{y \in \mathcal{C}_{TSP} : \pi y = \pi_0\}$ is a proper face of $\mathcal{C}_{TSP}$ and $\pi x^k \leq \pi_0$ is a valid m-PSP inequality, then $F^k = \{x \in \mathcal{C}_{PSP} : \pi x^k = \pi_0\}$ is a proper face of $\mathcal{C}_{PSP}$.*

Proof. Let y be a TSP solution satisfying $\pi y < \pi_0$ (such a solution exists because F is a proper face of $\mathcal{C}_{TSP}$). Using Lemma 1 we can construct an m-PSP solution with $x^k = y$. Thus $\pi x^k < \pi_0$ and hence $F^k \neq \mathcal{C}_{PSP}$. Let y' be another TSP solution satisfying $\pi y' = \pi_0$ (such a solution exists because F is a proper face of $\mathcal{C}_{TSP}$). Using Lemma 1 we can construct an m-PSP solution with $x^k = y'$. Thus $\pi x^k = \pi_0$ and hence $F^k \neq \emptyset$. Therefore F^k is a proper face of $\mathcal{C}_{PSP}$. □

Theorem 3. *If $F = \{y \in \mathcal{C}_{TSP} : \pi y = \pi_0\}$ is a facet of $\mathcal{C}_{TSP}$ and F^k is a proper face of $\mathcal{C}_{PSP}$, then F^k is a facet of $\mathcal{C}_{PSP}$.*

Proof. The proof is based on Theorem 3.6 of Chapter I.4 of Nemhauser and Wolsey [12]. Let $Ay = b$ define the equality constraints of $\mathcal{C}_{TSP}$ (i.e., the n degree constraints). If $\lambda y = \lambda_0$ for all $y \in F$, then $(\lambda, \lambda_0) = (\alpha\pi + \beta A, \alpha\pi_0 + \beta b)$ for some $\alpha \in \mathbb{R}$ and some $\beta \in \mathbb{R}^n$. Let $x \in \mathcal{C}_{PSP}$ be such that $x^k = y$. Then $\pi x^k = \pi_0$ and $Ax^k = b$. Now assume $\lambda' x^k = \lambda'_0$. Setting $\alpha' = \alpha$ and $\beta' = \beta$ so that $\lambda' = \lambda$, we have $\lambda'_0 = \lambda_0 = \alpha\pi_0 + \beta b$. Thus $\lambda' x^k = \lambda'_0 \implies (\lambda', \lambda'_0) = (\alpha\pi + \beta A, \alpha\pi_0 + \beta b)$ for some $\alpha \in \mathbb{R}$ and for some $\beta \in \mathbb{R}$. It follows that F^k is a facet of $\mathcal{C}_{PSP}$. □

A direct consequence of these theorems is that if $F = \{y \in \mathcal{C}_{TSP} : \pi y = \pi_0\}$ is a facet of $\mathcal{C}_{TSP}$ and if $\pi x^k \leq \pi_0$ is a valid m-PSP inequality, then $F^k = \{x \in \mathcal{C}_{PSP} : \pi x^k = \pi_0\}$ is a facet of $\mathcal{C}_{PSP}$. In particular subtour elimination constraints, 2-matching constraints, comb inequalities and clique tree inequalities are facet defining for the m-PSP because they are facet defining for the TSP [7].

3 2-index formulation

A natural relaxation of the 3-index model is obtained by aggregating the x_{ek} variables as $x_e = \sum_{k=1}^{m} x_{ek}$. Eliminating the now redundant constraints (4) and summing up the degree and subtour elimination constraints over k, we arrive at the following 2-index formulation:

$$\text{(2-index)} \qquad \text{minimize} \sum_{e \in E} c_e x_e \tag{6}$$

subject to

$$\sum_{e \in \delta(i)} x_e = 2m \qquad (i \in V) \quad (7)$$

$$\sum_{e \in E(S)} x_e \leq m(|S| - 1) \qquad (S \subset V, 2m + 1 \leq |S| \leq n - (2m + 1)) \quad (8)$$

$$x_e \in \{0, 1\} \qquad (e \in E). \quad (9)$$

As shown by Duchenne, Laporte and Semet [6], the 2-index model is only a relaxation of the m-PSP: some of its solutions are infeasible for the m-PSP. This is the case of the graph depicted in Figure 1 where all edge values are equal to 1. This graph describes a feasible 2-index solution, but it is not Hamiltonian [2]. As a result, any valid inequality for the 2-index model is valid for the m-PSP but the reverse is not true. However, the 2-index formulation provides a good starting point for an exact branch-and-cut algorithm.

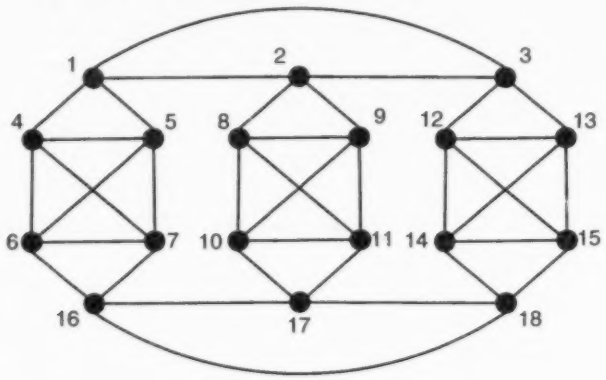


Figure 1: Non-Hamiltonian graph feasible for 2-index with $m = 2$

Theorem 4. *The strong 2-matching inequalities*

$$\sum_{e \in E(H)} x_e + \sum_{e \in T} x_e \leq m|H| + (|T| - 1)/2, \quad (10)$$

where $H \subset V$, $T \subset E$, $|T| \geq 3$ and odd, $|\{i,j\} \cap H| = 1$ for all $(i,j) \in T$ and $\{i,j\} \cap \{h,l\} = \emptyset$ for all $(i,j),(h,l) \in T$, $(i,j) \neq (h,l)$, are valid for the m-PSP.

Proof. See [6]. □

Theorem 5. *The weak 2-matching inequalities*

$$\sum_{e \in E(H)} x_e + \sum_{e \in T} x_e \leq m|H| + m(|T| - 1)/2, \quad (11)$$

where $H \subset V$, $T \subset E$, $|T| \geq 3$ and odd, $|\{i,j\} \cap H| = 1$ for all $(i,j) \in T$ and $\{i,j\} \cap \{h,l\} = \emptyset$ for all $(i,j),(h,l) \in T$, $(i,j) \neq (h,l)$, are valid for the m-PSP.

Proof. Constraints (11) are derived by summing up over k the following 2-matching inequalities which are valid for 3-index :

$$\sum_{e\in E(H)} x_{ek} + \sum_{e\in T} x_{ek} \leq |H| + (|T|-1)/2, \tag{12}$$

where $H \subset V$, $T \subset E$, $|T| \geq 3$ and odd, $|\{i,j\}\cap H| = 1$ for all $(i,j) \in T$ and $\{i,j\}\cap\{h,l\} = \emptyset$ for all $(i,j),(h,l) \in T$, $(i,j) \neq (h,l)$ and $k = 1,\ldots,m$. Since the right-hand side of (11) is greater than or equal to the right-hand side of (10), it follows that (11) is dominated by (10) and therefore valid for the m-PSP. □

Theorem 6. *The weak comb inequalities*

$$\sum_{e\in E(H)} x_e + \sum_{\ell=1}^{s} \sum_{e\in E(T_\ell)} x_e \leq m|H| + m\sum_{\ell=1}^{s}(|T_\ell|-1) - (ms+1)/2, \tag{13}$$

where $H, T_1,\ldots,T_s \subset V$, $s \geq 3$ and odd, $H\cap T_\ell \neq \emptyset$ and $T_\ell \setminus H \neq \emptyset$ $(\ell = 1,\ldots,s)$, $T_h \cap T_\ell = \emptyset$ $(h,\ell = 1,\ldots,s,\ h \neq \ell)$, and m odd, are valid for the m-PSP.

Proof. See [6]. □

Theorem 7. *The strong comb inequalities*

$$\sum_{e\in E(H)} x_e + \sum_{\ell=1}^{s} \sum_{e\in E(T_\ell)} x_e \leq m|H| + m\sum_{\ell=1}^{s}(|T_\ell|-1) - m(s+1)/2, \tag{14}$$

where $H, T_1,\ldots,T_s \subset V$, $s \geq 3$ and odd, $H\cap T_\ell \neq \emptyset$ and $T_\ell \setminus H \neq \emptyset$ $(\ell = 1,\ldots,s)$, and $T_h \cap T_\ell = \emptyset$ $(h,\ell = 1,\ldots,s,\ h \neq \ell)$, are valid for the m-PSP.

Proof. Constraints (14) are derived by summing up over k the following comb inequalities which are valid for the m-PSP :

$$\sum_{e\in E(H)} x_{ek} + \sum_{\ell=1}^{s} \sum_{e\in E(T_\ell)} x_{ek} \leq |H| + \sum_{\ell=1}^{s}(|T_\ell|-1) - (s+1)/2, \tag{15}$$

where $H, T_1,\ldots,T_s \subset V$, $s \geq 3$ and odd, $H\cap T_\ell \neq \emptyset$ and $T_\ell \setminus H \neq \emptyset$ $(\ell = 1,\ldots,s)$, $T_h \cap T_\ell = \emptyset$ $(h,\ell = 1,\ldots,s,\ h \neq \ell)$ and $k = 1,\ldots,m$. □

It is interesting to observe that contrary to what happened for the 2-matching inequalities, the comb inequalities obtained by aggregating the 3-index constraints are now stronger than those provided by Theorem 6. Moreover they are valid for odd and even values of m. While constraints (14) are valid for m-PSP, they are not valid for 2-index. Consider for example the graph of Figure 1 and let $H = \{1,2,3\}$, $T_1 = \{1,4,5,6,7\}$, $T_2 = \{2,8,9,10,11\}$ and $T_3 = \{3,12,13,14,15\}$. Then the left-hand side of constraints (14) is 33 while the right-hand side of constraints (13) is 32. This constraint is therefore violated. In other words, constraints (14) eliminate a feasible 2-index solution for this graph.

Theorem 8. *The weak clique tree inequalities*

$$\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + \sum_{j=1}^{s}\sum_{e\in E(T_j)} x_e \leq m\sum_{i=1}^{r}|H_i| + m\sum_{j=1}^{s}(|T_j| - t_j) - m\frac{s+1}{2} + \frac{mr+1}{2}, \quad (16)$$

where $H_1, \ldots, H_r, T_1, \ldots, T_s \subset V$, $H_i \cap H_j = \emptyset$ *for all* $i, j = 1, ..., r, i \neq j$, $|\{i : H_i \cap T_j \neq \emptyset\}| \geq 1$ *for all* $j = 1, 2, ..., s$, $T_i \cap T_j = \emptyset$ *for all* $1 \leq i < j \leq s$, $|\{j : H_i \cap T_j \neq \emptyset\}| \geq 3$ *and odd for all* $i = 1, ..., r$, $T_j \setminus \cup_{i=1}^{r} H_i \neq \emptyset$ *for all* $j = 1, ..., s$, *the associated hypergraph is connected, if* $H_i \cap T_j \neq \emptyset$, *it is a disconnecting set of the hypergraph,* t_j *is the number of handles* H_i $(i = 1, \ldots, r)$ *intersecting tooth* T_j *for all* $j = 1, ..., s$, m *and* r *odd, are valid for the* m*-PSP.*

Proof. Summing up constraints (7) over all vertices of $H_i (i = 1, ..., r)$ yields

$$2\sum_{e\in E(H_i)} x_e + \sum_{e\in\delta(H_i)} x_e = 2m|H_i|. \quad (17)$$

Summing up these equalities over all $i = 1, ..., r$, we obtain :

$$2\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + \sum_{i=1}^{r}\sum_{e\in\delta(H_i)} x_e = 2m\sum_{i=1}^{r}|H_i|. \quad (18)$$

However, since the T_j are disjoint, we have:

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{e\in\delta(H_i)\cap E(T_j)} x_e \leq \sum_{i=1}^{r}\sum_{e\in\delta(H_i)} x_e, \quad (19)$$

which yields the inequality:

$$2\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{e\in\delta(H_i)\cap E(T_j)} x_e \leq 2m\sum_{i=1}^{r}|H_i|. \quad (20)$$

Subtour elimination constraints for T_j et $T_j \setminus \cup_{i=1}^{r} H_i$, for all $j = 1, 2, ..., s$, and $T_j \cap H_i$ (for all $i = 1, ..., r$ and $j = 1, ..., s$ such that $T_j \cap H_i \neq \emptyset$) yield the inequalities

$$\sum_{e\in E(T_j)} x_e \leq m(|T_j| - 1), \quad (21)$$

$$\sum_{e\in E(T_j\setminus\cup_{i=1}^{r} H_i)} x_e \leq m(|T_j \setminus \cup_{i=1}^{r} H_i| - 1), \quad (22)$$

and

$$\sum_{e\in E(T_j\cap H_i)} x_e \leq m(|T_j \cap H_i| - 1). \quad (23)$$

Summing up inequality (20), inequalities (21) and (22) over all $j = 1, ..., s$, and inequalities (23) over all pairs (i, j) such that $i = 1, ..., r$, $j = 1, ..., s$ and $T_j \cap H_i \neq \emptyset$, yields:

$$2\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + \sum_{j=1}^{s}\sum_{e\in E(T_j)} x_e$$
$$+\sum_{j=1}^{s}\left[\sum_{e\in E(T_j\setminus\cup_{i=1}^{r}H_i)} x_e + \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{e\in\delta(H_i)\cap E(T_j)} x_e + \sum_{i=1,...,r \text{ and } T_j\cap H_i\neq\emptyset}\sum_{e\in E(T_j\cap H_i)} x_e\right]$$
$$\leq 2m\sum_{i=1}^{r}|H_i| + m\sum_{j=1}^{s}(|T_j|-1)$$
$$+\sum_{j=1}^{s}\left[m\left(|T_j\setminus\cup_{i=1}^{r}H_i|-1\right) + m\left(\sum_{i=1,...,r \text{ and } T_j\cap H_i\neq\emptyset}(|T_j\cap H_i|-1)\right)\right]. \tag{24}$$

Therefore,

$$2\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + 2\sum_{j=1}^{s}\sum_{e\in E(T_j)} x_e$$
$$\leq 2m\sum_{i=1}^{r}|H_i| + m\sum_{j=1}^{s}(|T_j|-1) + m\sum_{j=1}^{s}(|T_j|-1-t_j), \tag{25}$$

which, combined with $\sum_{j=1}^{s} t_j = s + r - 1$, yields

$$2\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + 2\sum_{j=1}^{s}\sum_{e\in E(T_j)} x_e$$
$$\leq 2m\sum_{i=1}^{r}|H_i| + 2m\sum_{j=1}^{s}(|T_j|-t_j) - ms - m + mr \tag{26}$$

and

$$\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + \sum_{j=1}^{s}\sum_{e\in E(T_j)} x_e$$
$$\leq m\sum_{i=1}^{r}|H_i| + m\sum_{j=1}^{s}(|T_j|-t_j) - m\frac{s+1}{2} + \frac{mr-1}{2} \tag{27}$$

whenever m and r are odd.

□

Theorem 9. *The strong clique tree inequalities*

$$\sum_{i=1}^{r}\sum_{e\in E(H_i)} x_e + \sum_{j=1}^{s}\sum_{e\in E(T_j)} x_e \leq m\sum_{i=1}^{r}|H_i| + m\sum_{j=1}^{s}(|T_j|-t_j) - m\frac{s+1}{2}, \tag{28}$$

where $H_1, \ldots, H_r, T_1, \ldots, T_s \subset V$, $H_i \cap H_j = \emptyset$ *for all* $i, j = 1, ..., r, i \neq j$, $|\{i : H_i \cap T_j \neq \emptyset\}| \geq 1$ *for all* $j = 1, 2, ..., s$, $T_i \cap T_j = \emptyset$ *for all* $1 \leq i < j \leq s$, $|\{j : H_i \cap T_j \neq \emptyset\}| \geq 3$ *and odd for all* $i = 1, ..., r$, $T_j \setminus \cup_{i=1}^{r} H_i \neq \emptyset$ *for all* $j = 1, ..., s$, *the associated hypergraph is connected, if* $H_i \cap T_j \neq \emptyset$, *it is a disconnecting set of the hypergraph, and* t_j *is the number of handles* H_i $(i = 1, \ldots, r)$ *intersecting tooth* T_j *for all* $j = 1, ..., s$, *are valid for the m-PSP.*

Proof. Constraints (28) are derived by summing up over k the following clique tree inequalities which are valid for the m-PSP :

$$\sum_{i=1}^{r} \sum_{e \in E(H_i)} x_{ek} + \sum_{j=1}^{s} \sum_{e \in E(T_j)} x_{ek} \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2}, \tag{29}$$

where $H_1, \ldots, H_r, T_1, \ldots, T_s \subset V$, $H_i \cap H_j = \emptyset$ for all $i, j = 1, ..., r, i \neq j$, $|\{i : H_i \cap T_j \neq \emptyset\}| \geq 1$ for all $j = 1, 2, ..., s$, $T_i \cap T_j = \emptyset$ for all $1 \leq i < j \leq s$, $|\{j : H_i \cap T_j \neq \emptyset\}| \geq 3$ and odd, for all $i = 1, ..., r$, $T_j \setminus \cup_{i=1}^{r} H_i \neq \emptyset$ for all $j = 1, ..., s$, the associated hypergraph is connected, if $H_i \cap T_j \neq \emptyset$, it is a disconnecting set of the hypergraph, t_j is the number of handles H_i $(i = 1, \ldots, r)$ intersecting tooth T_j for all $j = 1, ..., s$, $k = 1, \ldots, m$. □

Again, constraints (28) are stronger than constraints (15). The strong clique tree inequalities may eliminate feasible 2-index solutions. Consider for example the graph of Figure 2 and let $H_1 = \{1, 2, 3\}$, $H_2 = \{16, 19, 20\}$, $T_1 = \{1, 4, 5, 6, 7, 16\}$, $T_2 = \{2, 8, 9, 10, 11, 17\}$, $T_3 = \{3, 12, 13, 14, 15, 18\}$, $T_4 = \{19, 22, 23, 24, 25, 34\}$ and $T_5 = \{20, 26, 27, 28, 29, 35\}$. Then the left-hand side of constraints (28) is 56 while the right-hand side of constraints (16) is 54.5. This constraint is therefore violated. In other words, constraints (28) eliminate a feasible 2-index solution.

We will now prove that the subtour elimination constraints (8) are facet defining for 2-index. Let $\mathcal{C}_{2-index}$ denote the convex hull of the set of feasible solutions of 2-index.

Theorem 10. $dim(\mathcal{C}_{2-index}) = n(n-3)/2$.

Proof. We follow the lines of the proof of Proposition 3.4 of Chapter II.2 of Nemhauser and Wolsey [12] for the TSP polytope. Let $Q = \{x \in \{0,1\}^{n(n-1)/2} : x \text{ satifies (7)}\}$. Equation system (7) defines a constraint matrix of rank n. Hence, $dim(conv(Q)) = n(n-1)/2 - n = (n(n-3)/2$. Since $\mathcal{C}_{2-index} \subseteq conv(Q)$, it follows that $dim(\mathcal{C}_{2-index}) \leq n(n-3)/2$.

To prove that $dim(\mathcal{C}_{2-index}) = n(n-3)/2$, it suffices to show that if the hyperplane $\pi x = \pi_0, \pi \neq 0$ contains the incidence vector of every solution, then $\pi x = \pi_0$ is a linear combination of the constraints (7). Consider the two solutions depicted in Figure 3. Let $j \in \{6, ..., n\}$ and P_{j5} be two edge disjoint chains between j and 5 through all of the vertices $\{6, ..., n\}$. Now consider the pair of solutions $T_j^1 = P_{j5} \cup \{(1, j), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (2, 5), (3, 4), (3, j), (4, 5)\}$ and $T_j^2 = P_{j5} \cup \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, j), (3, 4), (3, j), (4, 5)\}$. Since T_j^1 et T_j^2 lie on the hyperplane $\pi x = \pi_0$, it follows that $\pi_{1j} + \pi_{25} =$

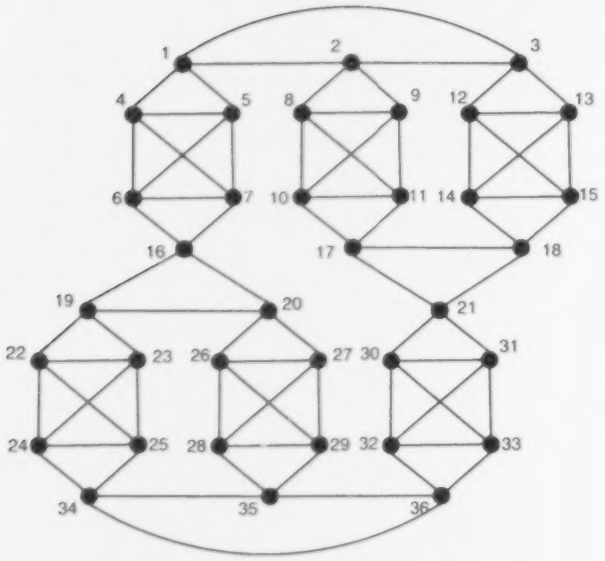


Figure 2: 2-index solution in which a strong clique tree inequality is violated($m = 2$)

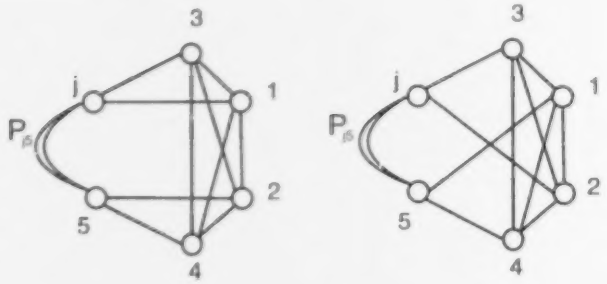


Figure 3: Two feasible 2-index solutions ($m = 2$) showing two edge disjoint chains between vertices 5 and j

$\pi_{2j} + \pi_{15}$, or $\pi_{2j} - \pi_{1j} = \pi_{25} - \pi_{15}$ for $j = 5, ..., n$. Let $\lambda_1 = \pi_{2j} - \pi_{1j}$ for $j = 5, ..., n$. Using an identical argument, we obtain $\lambda_i = \pi_{i+1,j} - \pi_{ij}$ for $i = 1, ..., n$ and $j > i + 1$, and $\lambda_i = \pi_{j,i+1} - \pi_{ji}$ for $i = 1, ..., n$ and $j < i$. Thus for any coefficient π_{ij}, we have:

$$
\begin{aligned}
\pi_{ij} &= (\pi_{ij} - \pi_{i-1,j}) + \pi_{i-1,j} \\
&= (\pi_{ij} - \pi_{i-1,j}) + \ldots + (\pi_{2j} - \pi_{1j}) + \pi_{1j} \\
&= \textstyle\sum_{t=1}^{i-1} \lambda_t + \pi_{1j} \\
&= \textstyle\sum_{t=1}^{i-1} \lambda_t + (\pi_{1j} - \pi_{1,j-1}) + \pi_{1,j-1} \\
&= \textstyle\sum_{t=1}^{i-1} \lambda_t + (\pi_{1j} - \pi_{1,j-1}) + \ldots + (\pi_{13} - \pi_{12}) + \pi_{12} \\
&= \textstyle\sum_{t=1}^{i-1} \lambda_t + \sum_{t=2}^{j-1} \lambda_t + \pi_{12} \\
&= u_i + u_j - u_2 + \pi_{12},
\end{aligned}
$$

where $u_i = \sum_{t=1}^{i-1} \lambda_t$ for $i > 1$ and $u_1 = 0$. Letting $\alpha = \pi_{12} - u_2$, we have:

$$
\begin{aligned}
\textstyle\sum_{e \in E} \pi_e x_e &= \textstyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \pi_{ij} x_{ij} \\
&= \textstyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (u_i + u_j + \alpha) x_{ij} \\
&= \textstyle\sum_{i=1}^{n} (u_i + \alpha/2) \left(\sum_{j<i} x_{ji} + \sum_{j>i} x_{ij} \right) \\
&= \textstyle\sum_{v \in V} (u_v + \alpha/2) \sum_{e \in \delta(\{v\})} x_e,
\end{aligned}
$$

which shows that (7) is a linear combination of degree constraints with $\pi_0 = 2m \sum_{v \in V} u_v + mn\alpha$. □

Theorem 11. *The subtour elimination constraints* (8) *are facet defining for* $\mathcal{C}_{2-index}$ *for* $n \geq 6$ *for all* m *and all* S *such that* $2m + 1 \leq |S| \leq n - (2m + 1)$.

Proof. We follow the lines of the proof of Proposition 3.5 of Chapter II.2 of Nemhauser and Wolsey [12] for the TSP subtour elimination constraints. We will prove the result for $n \geq 6$ and $|S| \geq 5$, where $S = \{1, \ldots, k\}$ and $k \leq \lfloor n/2 \rfloor$. Let $A^{=}x = b^{=}$ define the equality constraints of $\mathcal{C}_{2-index}$ (i.e., the n degree constraints) and $\pi x \leq \pi_0$ define the subtour elimination constraint (8). First observe that if (λ, λ_0) is a solution of $\lambda x^{T_i} = \lambda_0$, where $\{T_i\}_{i=1}^{n(n-3)/2}$ is a set of m-PSP solutions that satisfy $\pi x^{T_i} = \pi_0$, there is a solution (λ', λ_0') with $\lambda' = \lambda + u^1 A^{=}$ where $\lambda'_{1j} = 1$ for $j = 2, \ldots, k$, $\lambda'_{23} = 1$ and $\lambda'_{1j} = 0$ for $j = k+1, \ldots, n$. To see this, we observe that the $n \times n$ node-edge incidence matrix

$$
B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ . \\ . \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_{12} & e_{13} & e_{23} & e_{14} & \ldots & e_{1n} \end{array} \\
\left(\begin{array}{cccc|ccc}
1 & 1 & 0 & 1 & \ldots & 1 \\
1 & 0 & 1 & 0 & \ldots & 0 \\
0 & 1 & 1 & 0 & \ldots & 0 \\
\hline
\multicolumn{3}{c|}{\multirow{4}{*}{O}} & \multicolumn{3}{c}{\multirow{4}{*}{I}} \\
& & & & & \\
& & & & & \\
& & & & &
\end{array}\right)
\end{array}
$$

is non-singular. Hence the appropriate n components of λ' can be fixed by solving the $n \times n$ system:

$$
u^1 B = (1 - \lambda_{12}, 1 - \lambda_{13}, 1 - \lambda_{23}, \ldots, 1 - \lambda_{1k}, -\lambda_{1,k+1}, \ldots, -\lambda_{1n}).
$$

We now show in the following series of steps that: $\lambda'_{ij} = 1$ if $i, j \in S$; $\lambda'_{ij} = 0$ if $i \in S, j \notin S$ and $\lambda'_{ij} = \beta$ if $i, j \notin S$. Consider the two solutions $T_1 = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup$

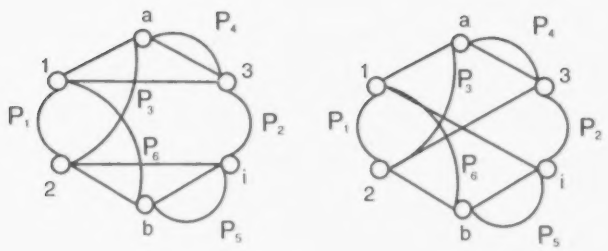


Figure 4: Two solutions of 2-index ($m = 2$)

$\{1,3\} \cup \{2,i\} \cup \{1,a\} \cup \{2,b\} \cup \{3,a\} \cup \{b,i\}$ and $T_2 = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup \{1,i\} \cup \{2,3\} \cup \{1,a\} \cup \{2,b\} \cup \{3,a\} \cup \{b,i\}$ of Figure 4, satisfying (8) at equality, and where $a, b \in V \setminus \{1,2,3,i\}$. The existence of such solutions can easily be established. We have $\lambda' x^{T_1} = \lambda' x^{T_2} = \lambda'_0$; thus, $\lambda'_{13} + \lambda'_{2i} = \lambda'_{1i} + \lambda'_{23}$. Therefore $\lambda'_{2i} = 1$ if $3 \leq i \leq k$ and $\lambda'_{2i} = 0$ if $k < i \leq n$.

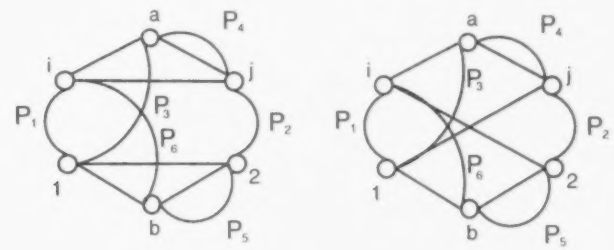


Figure 5: Two solutions of 2-index ($m = 2$)

The remaining cases are similar. Suppose $3 \leq i \leq k$ and consider two solutions $T_1 = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup \{i,j\} \cup \{1,2\} \cup \{a,i\} \cup \{1,b\} \cup \{a,j\} \cup \{2,b\}$ and $T_2 = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup \{2,i\} \cup \{1,j\} \cup \{a,i\} \cup \{1,b\} \cup \{a,j\} \cup \{2,b\}$ of Figure 5 where $a, b \in V \setminus \{1,2,i,j\}$. We can easily verify the existence of such solutions that satisfy constraints (8) at equality. So we have $\lambda' x^{T_1} = \lambda' x^{T_2} = \lambda'_0$; thus $\lambda'_{12} + \lambda'_{ij} = \lambda'_{2i} + \lambda'_{1j}$, and therefore $\lambda'_{ij} = 1$ if $i < j \leq k$ and $\lambda'_{ij} = 0$ if $k < j \leq n$.

The final case involves $p, q, r \notin S$ and solutions $T_1 = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup \{1,p\} \cup \{q,r\} \cup \{1,a\} \cup \{b,q\} \cup \{a,p\} \cup \{b,r\}$ and $T_2 = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup \{1,r\} \cup \{p,q\} \cup \{1,a\} \cup \{b,q\} \cup \{a,p\} \cup \{b,r\}$ of Figure 6 where $a, b \in V \setminus \{1,p,q,r\}$. We can easily verify the existence of such solutions that satisfy constraints (8) as equalities.

Therefore $\lambda' x^{T_1} = \lambda' x^{T_2} = \lambda'_0$ and $\lambda'_{1p} + \lambda'_{qr} = \lambda'_{1r} + \lambda'_{pq}$. It follows that $\lambda'_{pq} = \beta$ if

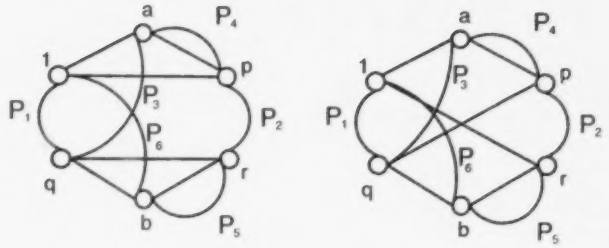


Figure 6: Two solutions of 2-index ($m = 2$)

$p, q \notin S$ where β is a constant. We have shown that $\lambda' x^{T_i} = \lambda'_0$ is of the form:

$$\sum_{e \in E(S)} x_e^{Ti} + \beta \sum_{e \in E(V \setminus S)} x_e^{Ti} = m(|S| - 1) + m\beta(|V \setminus S| - 1)$$

for any x^{T_i} satisfying (8) as an equality. Now defining $u^2 \in R^n$ by $u_i^2 = \frac{\beta}{2}$ if $i \in S$ and $u_i^2 = -\frac{\beta}{2}$ if $i \in V \setminus S$. The equality $(\lambda' + u^2 A^=) x^{T_i} = (\lambda'_0 + u^2 b^=)$ is then of the form:

$$\begin{aligned} (1+\beta) \textstyle\sum_{e \in E(S)} x_e^{Ti} &= m(|S| - 1) + m\beta(|V \setminus S| - 1) + m\beta|S| - m\beta|V \setminus S| \\ &= m(1+\beta)(|S| - 1). \end{aligned}$$

Therefore $(1 + \beta)\pi x^{T_i} = (1 + \beta)\pi_0$, and $(\lambda, \lambda_0) = ((1 + \beta)\pi - (u^1 + u^2)A^=, (1 + \beta)\pi_0 - (u^1 + u^2)b^=)$. Hence, subtour elimination constraints (8) define facets of $\mathcal{C}_{2-index}$. □

4 Algorithms

As shown by Duchenne, Laporte and Semet [6], the m-PSP can readily be solved applying the following branch-and-cut algorithm to 3-index.

3-index algorithm

The Krarup heuristic is applied to identify an initial feasible solution. A first subproblem is defined by (2) and $0 \leq x_{ek} \leq 1$ $(e \in E, k = 1, \ldots, m)$, and is solved in a branch-and-cut fashion by introducing violated subtour elimination, 2-matching and comb inequalities. When a solution is fractional and no violated constraint can be identified branching is performed according to a depth-first criterion on the fractional x_{ek} variable closest to 0.5. Using a depth-first criterion leads to a faster identification of a feasible solution.

However, a similar process cannot be immediately applied to 2-index because a solution to this model may be infeasible for the m-PSP. In Section 4.1 we show how to solve the *feasibility problem*, i.e. deciding whether a 2-index solution contains a feasible m-PSP solution. Section 4.2 is devoted to the *optimality problem*, i.e. determining a feasible and optimal m-PSP solution using 2-index.

4.1 Solving the feasiblity problem

Solving 2-index yields a *2m-regular graph* $\tilde{G} = (V, \tilde{E})$, i.e., a connected graph in which all vertives have degree $2m$, such as the graph of Figure 1. A graph is said to be *m-PSP feasible* if and only if it contains a feasible m-PSP solution. Determining whether $\tilde{G}$ is m-PSP feasible can be achieved heuristically or exactly.

Applying the Krarup heuristic

Let $z = \sum_{e \in \tilde{E}} c_e$. Applying the Krarup heuristic to $\tilde{G}$ either yields a feasible m-PSP solution of cost z or terminates without such a solution. In the latter case, if the optimal cost of the first TSP solution found by Krarup heuristic exceeds z/m, then no m-PSP solution exists on $\tilde{G}$.

Applying the 3-index algorithm

In the event where the Krarup algorithm fails, the 3-index algorithm can be applied to $\tilde{G}$ in order to determine whether this graph contains an exact m-PSP solution. In practice it is not always necessary to run this algorithm fully because it can stop at the first feasible solution.

Decomposition technique

When $\tilde{G}$ is large it may be advantageous to use a decomposition technique. We have developed the following scheme.

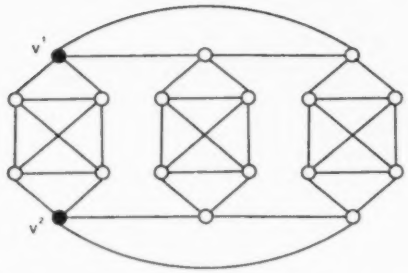
Step 1 An *articulation pair* is a subset $\{v^1, v^2\}$ of V such that removing v^1, v^2 and their incident edges yields two disconnected subgraphs $\tilde{G}_1^-$ and $\tilde{G}_2^-$ (Figure 7 a, b). Identifying v^1 and v^2 can easily be achieved in $O(n^3)$ time by applying Kleitman's algorithm [8], for example.

Step 2 The two vertices of the articulation pair are then reintroduced in $\tilde{G}_1^-$ and $\tilde{G}_2^-$, together with the edges connecting them to the other vertices of these graphs. This yields two augmented disconnected subgraphs $\tilde{G}_1$ and $\tilde{G}_2$ (Figure 7 c).

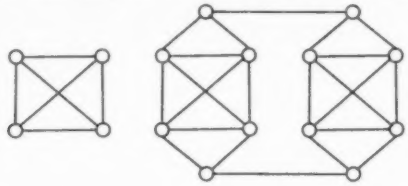
Step 3 In each of $\tilde{G}_1$ and $\tilde{G}_2$, artificial edges are introduced between v^1 and v^2 to bring their degree equal to $2m$. This yields two $2m$-regular graphs $\tilde{G}_1^+$ and $\tilde{G}_2^+$.

Theorem 12. *A 2m-regular graph $\tilde{G}$ is m-PSP feasible if and only if it decomposes into two m-PSP feasible graphs $\tilde{G}_1^+$ and $\tilde{G}_2^+$ defined by the above decomposition technique.*

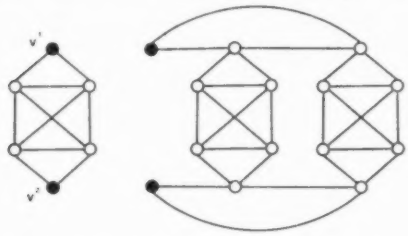
Proof. The graph $\tilde{G}$ contains a Hamiltonian cycle if and only if there exist two paths between v^1 and v^2 such that passing through all the vertices of $\tilde{G}$. Since $\tilde{G}_1$ and $\tilde{G}_2$ are disjoint, these paths exist if and only if there exist a Hamiltonian path between v^1 and v^2 in $\tilde{G}_1$ and another in $\tilde{G}_2$. By construction, there exists a Hamiltonian path between v^1 and v^2 in $\tilde{G}_1$ (resp. $\tilde{G}_2$) if and only if there exists a Hamiltonian cycle in $\tilde{G}_1^+$ (resp. $\tilde{G}_2^+$). Consequently, $\tilde{G}$ contains m Hamiltonian cycles if and only if $\tilde{G}_1^+$ and $\tilde{G}_2^+$ contain m Hamiltonian cycles. Moreover the cycles of $\tilde{G}$ are edge disjoint if and only if the cycles of



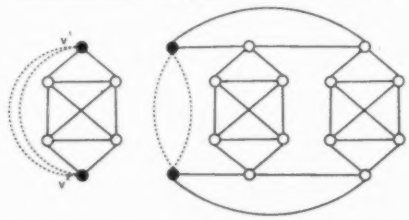
a) Graph $\tilde{G}$ showing an articulation pair (black vertices)



b) Two disconnected subgraphs $\tilde{G}_1^-$ and $\tilde{G}_2^-$



c) Two subgraphs $\tilde{G}_1$ and $\tilde{G}_2$ of $\tilde{G}$ obtained by reintroducing the articulation pair and the incident edges into $\tilde{G}_1$ and $\tilde{G}_2$



d) Reintroducing artificial edges (dotted) into $\tilde{G}_1$ and $\tilde{G}_2$ to obtain two $2m$-regular graphs $\tilde{G}_1^+$ and $\tilde{G}_2^+$

Figure 7: Decomposition technique

$\tilde{G}_1^+$ and of $\tilde{G}_2^+$ are edge disjoint. Therefore, $\tilde{G}$ contains m edge disjoint Hamiltonian cycles if and only if $\tilde{G}_1^+$ and $\tilde{G}_2^+$ contain m edge disjoint Hamiltonian cycles. □

¿From an algorithmic point of view Theorem 12 provides a simplification of the original feasibility problem because the Krarup heuristic or the 3-index algorithm can now be

applied to smaller graphs. In particular, if $\tilde{G}$ is not m-PSP feasible, this can be determined as soon as either $\tilde{G}_1^+$ or $\tilde{G}_2^+$ is shown not to be m-PSP feasible. It is also clear that a similar decomposition of $\tilde{G}$ into p subgraphs $\tilde{G}_1^+, \ldots, \tilde{G}_p^+$ can sometimes be determined by identifying more than one articulation pair in $\tilde{G}$.

4.2 Solving the optimality problem

Duchenne, Laporte and Semet [6] have developed a basic branch-and-cut algorithm based on 2-index.

Basic 2-index algorithm

The basic 2-index algorithm is similar to the 3-index algorithm. It solves 2-index but without 2-matching and comb inequalities. This yields a $2m$-regular graph $\tilde{G} = (V, \tilde{E})$ on which the feasibility problem is solved by applying the Krarup heuristic and the 3-index algorithm if necessary. If no feasible m-PSP solution is identified by either algorithm, the basic 2-index algorithm then terminates without a proven optimal m-PSP solution.

We have developed an improved branch-and-cut algorithm by adding several new features to the basic 2-index algorithm.

1) When solving 2-index we generate the strong 2-matching and comb inequalities.

2) Branching is performed according to a best first criterion, i.e., on the subproblem with the smallest lower bound value.

3) When applying the 3-index algorithm to solve the feasibility problem, we use the objective "minimize $\sum_{e \in \tilde{E}} c_e x_{e1}$". With this objective the 3-index algorithm converges faster toward a feasible solution, if one exists, than if a pure decidability problem has to be solved. In particular, active subproblems whose objective value exceeds z/m are fathomed and several symmetric solutions are disregarded.

4) Instead of solving the feasibility problem after the completion of the branch-and-cut tree, we now solve it at each node corresponding to a non-dominated integer solution. In addition, an attempt is made to solve the feasibility problem by means of the decomposition technique. If no articulation pair can be identified, the feasibility problem is solved on $\tilde{G}$. Whenever it can be shown that $\tilde{G}$ is not m-PSP feasible, a feasibility cut is added to the subproblem in order to prevent $\tilde{G}$ from reoccuring. This cut is:

$$\sum_{e \in \tilde{E}} x_e \leq |\tilde{E}| - 2, \text{ or } \sum_{e \in \tilde{E}_1} x_e \leq |\tilde{E}_1| - 1, \text{ or } \sum_{e \in \tilde{E}_2} x_e \leq |\tilde{E}_2| - 1,$$

depending on whether the m-PSP infeasibility was detected in $\tilde{G} = (V, \tilde{E})$, in $\tilde{G}_1 = (V_1, \tilde{E}_1)$ or in $\tilde{G} = (V_2, \tilde{E}_2)$. In addition, if an articulation pair was identified and one of $\tilde{G}_1$ or $\tilde{G}_2$ was m-PSP infeasible, then an attempt is made to identify another articulation pair (v^3, v^4) and subgraphs $\tilde{G}_3$, $\tilde{G}_4$ on which to check m-PSP feasibility.

¿From Theorem 12, either $\tilde{G}_3$ or $\tilde{G}_4$ must be m-PSP infeasible, and therefore at least one additional feasibility cut can be generated. This procedure can be reapplied several times.

The improved 2-index algorithm can now be described.

Improved 2-index algorithm

Step 1 (First subproblem) Determine a first m-PSP solution of cost $\overline{z}$ by means of the Krarup heuristic. Create a first subproblem defined by (7) and $0 \leq x_e \leq 1$ ($e \in E$). Insert this subproblem in a list.

Step 2 (Termination check) If the list is empty, stop with the optimal solution. Otherwise select a subproblem from the list according to a best first criterion.

Step 3 (Subproblem solution) Solve the subproblem and let z be its solution value. If $z \geq \overline{z}$, go to Step 2.

Step 4 (Cutting) Check in this order violations of subtour elimination constraints (8), strong 2-matching inequalities (10), and strong comb inequalities (14). If violated constraints can be identified, introduce them in the subproblem and go to Step 3.

Step 5 (Feasibility problem) If the solution is integer solve the feasibility problem. If the solution is m-PSP feasible, set $\overline{z} := z$, record the solution and go to Step 2. Otherwise generate feasibility cuts, introduce them in the subproblem and go to Step 3.

Step 6 (Branching) Create two subproblems by branching on the fractional variable closest to 0.5. Ties are broken by branching on the variable with the largest cost.

In this algorithm, violated subtour elimination constraints are detected by means the exact flow separation procedure of Padberg and Grötschel [13]. The violated 2-matching and comb inequalities were generated by slightly adapting the heuristic procedure proposed by Naddef and Thienel [11]. Very few comb inequalities were generated in our experiments. As a result we did not generate clique tree inequalities which are more difficult to separate and unlikely to be binding.

5 Computational results

The algorithms just described were coded in C^{++} and run to a 3.4 GHz Pentium IV Extreme Edition. All integer linear programs were solved with CPLEX 9.1 and we used Concorde [1] to solve the TSPs.

Tests were performed on randomly generated Euclidean instances obtained by first generating n points in the $[0, 100]^2$ square according to a discrete uniform distribution and by computing the c_es as Euclidean distances. Five instances were generated for $n =$

$10, \ldots, 190$ and $m = 2, \ldots, 10$. We also performed tests on several TSPLIB [14] instances with $21 \leq n \leq 280$.

Computational results are summarized in Tables 1 to 4. The table headings are as follows:

NAME:	name of the TSPLIB instance;
n:	number of vertices in G;
m:	number of edge-disjoint cycles in the solution;
OPT:	optimal m-PSP solution value;
SUCC:	success rate of the algorithm: number of optimal solutions found within the time limit divided by the number of attempted instances;
SROOT:	success rate at the root of the search tree before branching;
SEC2:	number of subtour elimination constraints generated in Step 4 of the improved 2-index algorithm;
SEC3:	total number of subtour elimination constraints generated in all the feasibility problems solved in Step 5 of the improved 2-index algorithm;
SC2:	number of strong comb inequalities (including the strong 2-matching inequalities which are a special case) generated in Step 4 of the improved 2-index algorithm;
SC3:	total number of comb inequalities (including the 2-matching inequalities which are a special case) generated in all the feasibility problems solved in Step 5 of the improved 2-index algorithm;
FP:	number of feasibility problems solved in the search tree;
FC:	number of feasibility cuts generated in the search tree (FC $\geq$ FP because more than one feasibility cut may be generated for a given feasibility problem);
NODES2:	number of nodes in the search tree of the improved 2-index algorithm;
NODES3:	total number of nodes in all feasibility problems solved in Step 5 of the improved 2-index algorithm;
s:	CPU time in seconds;
%sFEAS:	% of the CPU time spent in the feasibility algorithms; when the CPU time is too small, this percentage is not computed.

All statistics are average over the number of succesful instances, i.e., instances that could be solved optimally within 3600 CPU seconds.

Table 1 indicate that the improved 2-index algorithm was capable of solving randomly generated Euclidean instances with $n \leq 190$ and $m \leq 2$. As can be expected, larger values

of m yield more difficult instances. As a rule the number of nodes in the search tree is relatively small and few valid inequalities are generated when the problem can be solved within the assigned time. However, when this is not the case, a larger number of strong inequalities are generated and these cut a lot of node in the search tree. As a result the lower bound increase quickly. Overall the main source of difficulty lies in the number of calls to the feasibility algorithm. In several cases this prevented us from solving the m-PSP within the prescribed time limit. Problems that could be solved required the solution of very few feasibility problems. For example one particular instance with $n = 50$ and $m = 2$ required 565 calls to the feasibility algorithm and still could not be solved within 3600 seconds. The last column of Tables 1 and 2 clearly shows that the largest portion of the computing time is spent in the feasibility algorithm. Table 2 shows that larger TSPLIB instances can be solved, but the behaviour of the algorithm is basically the same for these instances.

Tables 3 and 4 provide comparisons between the 3-index, the basic 2-index and the improved 2-index algorithms. These tables clearly show the superiority of the latter algorithm both in terms of the success rate and of computing times.

6 Conclusions

We have developed new valid inequalities, polyhedral results and a new 2-index branch-and-cut algorithm for the *m-Peripatetic Salesman Problem* introduced by Krarup [9]. Our results demonstrate that the proposed 2-index algorithm clearly outperforms the basic version developed by Duchenne, Laporte and Semet [6], which was itself superior to the branch-and-bound algorithm of De Kort [5]. The success of our algorithm lies partly in the use of a relaxed 2-index model which often provides a feasible m-PSP solution. The most time consuming part of our algorithm is the feasibility algorithm called in Step 5. With respect to the basic 2-index algorithm the improved version generates more valid inequalities which reduces the size of the search tree and, indirectly, the number of calls to the feasibility algorithm. Our improved 2-index is the first ever a method capable of solving Euclidean m-PSP instances containing up to $n = 280$ vertices with $m = 2$, $n = 80$ vertices with $m = 5$ and $n = 50$ vertices with $m = 10$. This represents more than twice the size of instances that could previously be solved.

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Table 1: Computational results for the improved 2-index algorithm: random Euclidean instances.

n	m	SROOT	SEC2	SEC3	SC2	SC3	NODES2	NODES3	FP	FC	SUCC	s(%sFEAS)
10	2	1	0.6	0	0	0	0.6	0	0.6	0	1	0
	3	1	0	7.8	0	0	0.6	0	0.6	0	1	0
	4	1	0	0	0	0	0	0	0	0	1	0
20	2	0.6	0.6	12	0	1.2	1.6	0.6	1	0	1	0
	3	0.8	0	34.2	0	4.2	1	1.4	0.8	0	1	0
	4	0.8	0	192	0	15.2	1.2	13.4	1	0	1	0
	5	1	0	346	0	57	1	32	1	0	1	2(100)
	6	1	0	822	0	246	1	68.2	1	0	1	11(100)
	7	1	0	1323	0	357	1	105.8	1	0	1	30(100)
	8	1	0	2427.2	0	632	1	150.8	1	0	1	77(100)
	9	1	0	1629	0	340.2	0.4	170.2	0.4	0	1	83(100)
30	2	1	2.2	9.2	0	0.8	0.8	170.6	0.8	0	1	0
	3	0.8	0.6	82.8	0	3.6	1.2	175.4	1	0	1	0
	4	1	0	380	0	75.2	1	199	1	0	1	5(100)
	5	0.6	0	997	0	206	1.4	251.4	1	0	1	27(100)
	6	1	0	1716	0	298.8	0.8	309.4	0.8	0	1	90(100)
	7	1	0	3955	0	709.8	1	397.6	1	0	1	344(100)
	8	0.6	0	3936	0	500.8	3.4	455.6	1	0	1	416(100)
	9	0.6	0	2833.2	0	266.4	2	478.2	1	0	1	129(100)
	10	0.4	0	4518	0	452	3	508.8	1	0	1	370(100)
40	2	0.2	3	64.8	0	6.8	7	525.4	1.8	1.8	1	1(90)
	3	0.6	0.4	177.6	0	18	3.2	539	1	0	1	2(100)
	4	0.8	0.4	758.4	0	126.4	1.6	586.8	1	0	1	30(100)
	5	0.6	0	2230	0	351.2	2	823.8	1	0	0.8	222(100)
	6	0.8	0	4782	0	564	1	937.2	1	0	0.8	877(100)
	7	0.6	0	2516.5	0	299.2	9.5	965.2	1	0	0.8	180(99)
	8	0.6	0	4569.6	0	372.8	1.4	823.4	1	0	1	728(100)
	9	0.8	0	5068.8	0	531	1.2	865	1	0	1	1148(100)
	10	0.6	0	2593.3	0	233.3	1	1453	1	0	0.6	281(100)
50	2	0.2	5.3	62	2	4.7	22	1459.3	1	0	0.6	2(67)
	3	0.6	0.2	351	0	25.8	7.4	903.6	1	0	1	10(97)
	4	0.8	0	1507.2	0	256.8	1.2	1017.6	1	0	1	202(100)
	5	1	0	3223	0	457	1	1127	1	0	1	552(100)
	6	0.4	0	4417.5	0	405	2.5	1500.2	1	0	0.8	869(100)
	7	0.6	0	3748.5	0	334.2	1.2	1535	1	0	0.8	543(100)
	8	0.6	0	4811.2	0	332.8	1.6	1263.8	1	0	1	1168(100)
	9	1	0	4368.6	0	293.4	1	1291.4	1	0	1	1119(100)
	10	0.8	0	5267.5	0	285	1	1641.2	1	0	0.8	1514(100)
60	2	0	7	1055.6	0	173.2	35.2	2668.8	8.2	8.6	1	110(88)
	3	0.2	0	535.2	0	42	2.6	2703	1	0	1	21(100)
	4	0	0	2644	0	288.8	5	2833.4	1	0	1	561(100)
	7	0.6	0	4984	0	501.7	1	4768	1	0	0.6	1380(100)
	8	0.6	0	5421.3	0	352	1	4821.7	1	0	0.6	2102(100)
	9	0.4	0	4405.5	0	279	1	7252.5	1	0	0.4	1494(100)
70	2	0.4	1.8	284.4	0	74.8	8.4	3202.8	1	0	1	64(92)
	3	0	0.4	733.8	0	91.2	5.2	3258.8	1	0	1	73(99)
	4	0.2	0	3010.7	0	293.3	7.7	5571.7	1	0	0.6	1080(100)
	5	0	0	3085	0	405	8	16764	1	0	0.2	690(100)
	6	0	0	5184	0	462	2	16824	1	0	0.2	1902(100)
	7	0	0	4070.5	0	192.5	5.5	8460.5	1	0	0.4	1490(100)
	8	0.4	0	3800	0	176	1	8489.5	1	0	0.4	1454(100)
	9	0.2	0	4383	0	108	1	17007	1	0	0.2	1960(100)
80	2	0	6.2	7874	0	840	194.2	5863.8	36	73	0.8	414(78)
	3	0.4	0.4	783.6	0	78.6	33.2	4751	1	0	1	115(97)
	4	0	0.3	2897.3	0	221.3	11.7	8043.7	1	0	0.6	1265(100)
	5	0.2	0	2075	0	170	1	24170	1	0	0.2	521(100)
	7	0	0	4788	0	182	2	24216	1	0	0.2	2372(100)
	8	0	0	4088	0	104	3	24245	1	0	0.2	2147(100)
90	2	0	4.5	1418.5	0.8	213.5	21.8	10361.2	4.8	10.8	0.8	753(91)
	3	0	0	980	0	100	37.3	13877.7	1	0	0.6	205(98)
	4	0.2	0	3236	0	294	3.5	20952	1	0	0.4	2206(100)
	6	0	0	5280	0	168	19	42000	1	0	0.2	3339(100)

n	m	SROOT	SEC2	SEC3	SC2	SC3	NODES2	NODES3	FP	FC	SUCC	s(%sFEAS)
100	2	0	5.7	356	0	70	4.3	833	1	0	0.6	332(91)
	3	0	0.6	1368.6	0	90.6	347.6	591	1	0	1	614(91)
110	2	0	6.5	612	0	74	27.5	1531	3	8.5	0.4	58(63)
	3	0.2	0.3	1515	0	121	10.7	1142	1	0	0.6	976(99)
120	2	0	4.5	766.5	0	69.5	13.8	1115	1	0	0.8	364(96)
	3	0	1	1635	0	97.5	2	2323	1	0	0.4	944(100)
130	2	0	8.5	665	0	73	35.5	2448	1	0	0.4	269(89)
	3	0	1	1868	0	121	12	1730	1	0	0.6	1396(99)
140	2	0	4.3	732	0	63.3	239	1826	1.3	1	0.6	373(72)
	3	0	0	1960.5	0	72	90.5	2863	1	0	0.4	1806(97)
150	3	0	2	2799	0	171	77	5860	1	0	0.2	3267(98)
160	2	0	9	1130	0	118	9	6074	1	0	0.2	839(98)
170	2	0	13	1306	0	72.7	200	2215	1	0	0.6	1549(78)
190	2	0	7	1884	0	90	165	6830	1	0	0.2	2694(90)

Table 2: Computational results for the improved 2-index algorithm: TSPLIB instances.

NAME(n)	m	SROOT	SEC2	SEC3	SC2	SC3	NODES2	NODES3	FP	FC	OPT	s(%sFEAS)
gr21	2	0	3	30	0	0	2	2	1	0	6900	0
	3	0	0	27	0	6	2	3	1	0	12486	0
	4	0	0	96	0	20	2	16	1	0	19076	0
	5	0	0	410	0	110	2	19	1	0	26651	3(100)
	6	0	0	1206	0	498	2	50	1	0	35621	26(100)
	7	0	0	889	0	175	2	25	1	0	45261	11(100)
	8	0	0	3712	0	776	2	78	1	0	55376	159(100)
	9	0	0	6336	0	1611	2	97	1	0	65736	478(100)
	10	1	0	0	0	0	1	0	0	0	76416	0
ulysses22	2	0	6	12	0	0	3	0	1	0	16554	0
	3	0	2	84	0	0	2	3	1	0	27676	0
	4	0	1	48	0	8	2	5	1	0	40535	0
	5	0	0	440	0	40	2	30	1	0	57472	3(100)
	6	0	0	1296	0	420	2	55	1	0	77194	31(100)
	7	0	0	2611	0	595	2	76	1	0	98258	92(100)
	8	0	0	3272	0	680	2	52	1	0	119709	113(100)
	9	0	0	5184	0	1827	2	51	1	0	141450	295(100)
	10	0	0	5950	0	1110	2	43	1	0	163438	272(100)
gr24	2	0	0	4	0	0	2	0	1	0	3147	0
	3	0	0	45	0	0	2	4	1	0	5614	0
	4	0	0	104	0	0	2	2	1	0	8435	0
	5	0	0	770	0	190	11	45	1	0	11665	13(100)
	6	0	0	1278	0	198	2	63	1	0	15364	35(100)
	7	0	0	2156	0	658	2	49	1	0	19544	72(100)
	8	0	0	2808	0	656	2	42	1	0	23996	105(100)
	9	0	0	2619	0	495	2	20	1	0	28631	70(100)
	10	0	0	3540	0	650	2	23	1	0	33368	122(100)
	11	1	0	0	0	0	1	0	0	0	38257	0
fri26	2	0	3	10	0	0	2	0	1	0	2218	0
	3	0	0	45	0	0	2	1	1	0	3974	0
	4	0	0	252	0	40	2	15	1	0	6080	1(100)
	5	0	0	935	0	130	2	43	1	0	8489	17(100)
	6	0	0	2352	0	528	2	69	1	0	11230	108(100)
	7	0	0	3850	0	567	2	91	1	0	14322	245(100)
	8	0	0	6000	0	1112	2	80	1	0	17586	526(100)
	9	0	0	5859	0	666	2	52	1	0	21007	402(100)
	10	0	0	2690	1	470	2	9	1	0	24538	65(100)
	11	0	0	5368	0	429	2	38	1	0	28145	379(100)
	12	1	0	0	0	0	1	0	0	0	31820	0(100)
swiss42	2	0	0	52	0	8	4	9	1	0	3100	0
	3	0	0	345	0	60	3	17	1	0	5541	4(100)
	4	0	0	932	0	268	2	48	1	0	8487	40(100)
	5	0	0	4405	0	995	2	161	1	0	11811	951(100)
	6	0	0	4722	0	492	2	102	1	0	15532	672(100)
	7	0	0	8260	0	742	7	131	1	0	19650	2084(100)
	8	0	0	5640	0	544	2	61	1	0	24142	969(100)
gr48	2	0	3	68	0	12	3	3	1	0	11816	0
	3	0	0	519	0	66	59	70	1	0	20738	26(88)
	4	0	0	1208	0	320	2	49	1	0	31615	70(100)
	5	0	0	2725	0	495	7	86	1	0	44331	317(100)
	6	0	0	7488	0	888	2	200	1	0	58596	2635(100)
hk48	2	0	3	100	0	2	7	9	1	0	27838	1(100)
	3	0	0	198	0	9	2	8	1	0	49306	1(100)
	4	0	0	704	0	80	2	43	1	0	75056	28(100)
	6	0	0	2556	0	264	5	50	1	0	138243	237(100)
	7	0	0	4151	0	637	2	45	1	0	174888	547(100)
	8	0	0	5640	0	184	4	64	1	0	214929	1253(100)
	9	0	0	3258	0	270	2	16	1	0	258234	362(100)
eil51	2	0	0	160	0	16	2	19	1	0	982	1(100)
	3	0	0	498	0	48	8	41	1	0	1737	19(100)
	4	0	0	888	0	156	4	47	1	0	2622	52(100)
	5	0	0	4150	0	605	3	130	1	0	3628	1237(100)

NAME(n)	m	SROOT	SEC2	SEC3	SC2	SC3	NODES2	NODES3	FP	FC	OPT	s(%sFEAS)
	6	0	0	4386	0	258	2	104	1	0	4745	862(100)
	7	0	0	3402	0	203	2	33	1	0	5958	450(100)
	8	0	0	2608	0	264	2	17	1	0	7261	372(100)
	9	0	0	4059	0	153	2	36	1	0	8666	1015(100)
brazil58	2	0	23	13368	0	968	3337	1434	156	647	59150	1203(9)
	3	0	3	381	0	36	2	14	1	0	111308	8(100)
	4	0	1	804	0	48	2	45	1	0	176802	49(100)
	5	0	1	4900	0	470	2	164	1	0	248702	1277(100)
	6	0	1	5460	0	336	2	108	1	0	326219	1336(100)
	7	0	1	2401	0	49	2	29	1	0	413164	275(100)
	8	0	0	5176	0	192	2	45	1	0	513394	1454(100)
	9	0	0	4896	0	198	2	39	1	0	625626	1284(100)
st70	2	0	14	4942	0	508	622	602	99	100	1619	300(47)
	4	0	0	4372	0	564	4	278	1	0	4340	3067(100)
eil76	2	0	0	96	0	14	11	13	1	0	1257	4(75)
	3	0	0	849	0	42	7	88	1	0	2141	146(99)
pr76	2	0	3	362	0	18	5	35	1	0	266395	19(100)
	3	0	0	957	0	78	9	95	1	0	485719	181(99)
	4	0	0	2228	0	336	6	98	1	0	738418	747(99)
gr96	3	0	1	768	0	54	34	61	1	0	234536	217(95)
rat99	2	0	3	226	0	30	32	21	1	0	2928	23(60)
	4	0	0	2224	0	144	6	112	1	0	7760	1074(99)
kroB100	2	0	9	1368	0	134	62	385	3	4	53600	209(88)
	3	0	2	1209	0	60	5	72	1	0	96691	301(99)
	4	0	0	3520	0	252	9	125	1	0	147224	2287(99)
kroC100	2	0	11	1308	0	48	220	177	5	58	51157	137(19)
	3	0	0	1722	0	300	26	395	1	0	92775	2050(99)
kroD100	2	0	5	172	0	2	4	24	1	0	51838	10(90)
	3	0	0	999	0	168	4	114	1	0	91200	221(100)
	4	0	0	3572	0	280	3	117	1	0	139652	1965(100)
kroE100	2	0	0	338	0	36	4	37	1	0	54183	23(100)
	3	0	0	888	0	78	127	53	1	0	96406	245(84)
eil101	2	0	0	308	0	18	5	28	1	0	1453	17(94)
	3	0	0	588	0	18	5	44	1	0	2468	97(98)
lin105	2	0	13	1552	0	180	120	695	7	36	36361	275(76)
pr107	2	0	32	138	0	6	2	9	1	0	98381	9(11)
	3	0	13	651	0	33	2	38	1	0	169298	142(97)
	4	0	3	4516	0	280	3	153	1	0	263332	3129(99)
gr120	2	0	4	224	0	30	13	23	1	0	16400	10(40)
pr124	3	0	13	882	0	75	3	41	1	0	265906	201(97)
	4	0	8	2812	0	232	2	106	1	0	412429	2298(99)
ch130	2	0	17	1008	0	142	190	2138	1	0	14854	2159(90)
pr136	2	0	7	294	0	26	2	34	1	0	220642	51(94)
	3	0	1	867	0	93	16	55	1	0	383804	411(97)
gr137	2	0	7	244	0	4	62	23	1	0	169628	73(38)
	3	0	2	1296	0	108	24	105	1	0	301768	1025(98)
kroA150	2	0	8	1290	0	52	29	136	1	0	64249	749(95)
	2	0	14	844	0	44	7	165	1	0	62710	503(97)
d198	2	0	26	460	0	24	39	49	1	0	35404	336(63)
kroA200	2	0	13	624	0	40	4	62	1	0	70871	311(93)
pr226	2	0	181	450	0	16	3	43	1	0	183537	866(20)
pr264	2	0	43	1340	0	62	3	155	1	0	122941	3142(97)
a280	2	0	3	1110	0	46	225	110	1	0	6317	3085(73)

Table 3: Success rate and computation time for the various algorithms: random Euclidean instances.

		3-index		basic 2-index		improved 2-index	
n	m	SUCC	SECONDS	SUCC	SECONDS	SUCC	SECONDS
10	2	1	0	1	0	1	0
	3	1	0	1	0	1	0
	4	1	1	1	0	1	0
20	2	1	11	1	1	1	0
	3	1	59	1	29	1	0
	4	1	220	1	219	1	0.2
	5	1	667	1	515	1	1.6
	6	1	1097	0.6	1207	1	11.2
	7	0	-	0.6	2052	1	30.2
	8	0	-	0.2	3318	1	77
	9	0	-	0	-	1	82.8
30	2	1	19	1	31	1	0
	3	0.8	511	1	597	1	0
	4	0.6	2166	0.8	1997	1	4.6
	5	0	-	0	-	1	26.6
	6	0	-	0	-	1	90.4
	7	0	-	0	-	1	343.6
	8	0	-	0	-	1	416
	9	0	-	0	-	1	129
	10	0	-	0	-	1	369.8
40	2	0.8	87	0.8	42	1	0.6
	3	0.6	1155	0.6	1230	1	1.6
	4	0	-	0	-	1	30.4
	5	0	-	0	-	0.8	222.5
	6	0	-	0	-	0.8	877
	7	0	-	0	-	0.8	179.8
	8	0	-	0	-	1	728.4
	9	0	-	0	-	1	1147.8
	10	0	-	0	-	0.6	281
50	2	0.2	158	0.4	104	0.6	2.3
	3	0	-	0	-	1	9.8
	4	0	-	0	-	1	202
	5	0	-	0	-	1	552.2
	6	0	-	0	-	0.8	868.8
	7	0	-	0	-	0.8	543.2
	8	0	-	0	-	1	1167.6
	9	0	-	0	-	1	1118.8
	10	0	-	0	-	0.8	1513.8
60	2	0	-	0.4	243	1	110.2
	3	0	-	0	-	1	21.4
	4	0	-	0	-	1	560.8
	7	0	-	0	-	0.6	1380.3
	8	0	-	0	-	0.6	2102
	9	0	-	0	-	0.4	1494
70	2	0	-	0.6	1691	1	64.4
	3	0	-	0	-	1	72.6
	4	0	-	0	-	0.6	1079.7
	5	0	-	0	-	0.2	690
	6	0	-	0	-	0.2	1902
	7	0	-	0	-	0.4	1489.5
	8	0	-	0	-	0.4	1453.5
	9	0	-	0	-	0.2	1960
80	2	0	-	0	-	0.8	414.5
	3	0	-	0	-	1	115.4
	4	0	-	0	-	0.6	1265
	5	0	-	0	-	0.2	521
	7	0	-	0	-	0.2	2372
	8	0	-	0	-	0.2	2147
90	2	0	-	0.2	2701	0.8	753
	3	0	-	0	-	0.6	205
	4	0	-	0	-	0.4	2206.5
	6	0	-	0	-	0.2	3339

		3-index		basic 2-index		improved 2-index	
n	m	SUCC	SECONDS	SUCC	SECONDS	SUCC	SECONDS
100	2	0	-	0	-	0.6	331.7
	3	0	-	0	-	1	614.4
110	2	0	-	0	-	0.4	58
	3	0	-	0	-	0.6	976
120	2	0	-	0	-	0.8	363.8
	3	0	-	0	-	0.4	944.5
130	2	0	-	0	-	0.4	269
	3	0	-	0	-	0.6	1395.7
140	2	0	-	0	-	0.6	372.7
	3	0	-	0	-	0.4	1805.5
150	3	0	-	0	-	0.2	3267
160	2	0	-	0	-	0.2	839
170	2	0	-	0	-	0.6	1549
190	2	0	-	0	-	0.2	2694

Table 4: Computation time for the various algorithms: TSPLIB instances.

		3-index	basic 2-index	improved 2-index
NAME(n)	m	SECONDS	SECONDS	SECONDS
gr21	2	1	1	0
	3	891	28	0
	4	158	287	0
	5	-	835	3
	6	1662	1464	26
	7	2525	1553	11
	8	-	-	159
	9	-	-	478
	10	-	-	0
ulysses22	2	6	2	0
	3	51	25	0
	4	296	288	0
	5	903	3326	3
	6	1564	1524	31
	7	-	2633	92
	8	-	-	113
	9	-	-	295
	10	-	-	272
gr24	2	14	1	0
	3	1246	149	0
	4	2963	741	0
	5	-	1392	13
	6	2825	-	35
	7	-	-	72
	8	-	-	105
	9	-	-	70
	10	-	-	122
	11	-	-	0
fri26	2	15	13	0
	3	122	114	0
	4	916	968	1
	5	-	2576	17
	6	-	-	108
	7	-	-	245
	8	-	-	526
	9	-	-	402
	10	-	-	65
	11	-	-	379
	12	-	-	0
swiss42	2	-	-	0
	3	-	-	4
	4	-	-	40
	5	-	-	951
	6	-	-	672
	7	-	-	2084
	8	-	-	969
gr48	2	2100	172	0
	3	-	-	26
	4	-	-	70
	5	-	-	317
	6	-	-	2635
hk48	2	-	48	1
	3	-	-	1
	4	-	-	28
	6	-	-	237
	7	-	-	547
	8	-	-	1253
	9	-	-	362
eil51	2	315	212	1
	3	-	-	19
	4	-	-	52
	5	-	-	1237

		3-index	basic 2-index	improved 2-index
NAME(n)	m	SECONDS	SECONDS	SECONDS
	6	-	-	862
	7	-	-	450
	8	-	-	372
	9	-	-	1015
brazil58	2	-	-	1203
	3	-	-	8
	4	-	-	49
	5	-	-	1277
	6	-	-	1336
	7	-	-	275
	8	-	-	1454
	9	-	-	1284
st70	2	-	-	300
	4	-	-	3067
eil76	2	-	-	4
	3	-	-	146
pr76	2	-	-	19
	3	-	-	181
	4	-	-	747
gr96	3	-	-	217
rat99	2	-	-	23
	4	-	-	1074
kroB100	2	-	-	209
	3	-	-	301
	4	-	-	2287
kroC100	2	-	-	137
	3	-	-	2050
kroD100	2	-	-	10
	3	-	-	221
	4	-	-	1965
kroE100	2	-	-	23
	3	-	-	245
eil101	2	-	-	17
	3	-	-	97
lin105	2	-	-	275
pr107	2	-	-	9
	3	-	-	142
	4	-	-	3129
gr120	2	-	-	10
pr124	3	-	-	201
	4	-	-	2298
ch130	2	-	-	2159
pr136	2	-	-	51
	3	-	-	411
gr137	2	-	-	73
	3	-	-	1025
kroA150	2	-	-	749
kroB150	2	-	-	503
d198	2	-	-	336
kroA200	2	-	-	311
pr226	2	-	-	866
pr264	2	-	-	3142
a280	2	-	-	3085